Modern C++ Programming

3. Basic Concepts II

Integral and Floating-point Types

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Integral Data Types

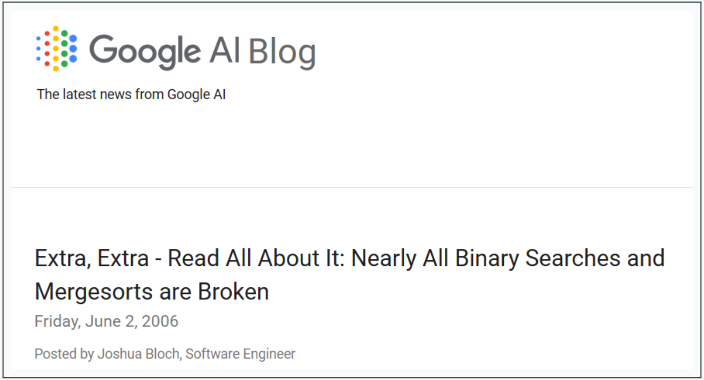
A Firmware Bug

> *"Certain SSDs have a firmware bug causing them to irrecoverably fail after exactly 32,768 hours of operation. SSDs that were put into service at the same time will fail simultaneously, so RAID won't help"*

`HPE SAS Solid State Drives - Critical Firmware Upgrade`

Via `twitter.com/martinkl/status/1202235877520482306?s=19`

Overflow Implementations



Google AI Blog

The latest news from Google AI

Extra, Extra - Read All About It: Nearly All Binary Searches and Mergesorts are Broken

Friday, June 2, 2006

Posted by Joshua Bloch, Software Engineer

Note: Computing the average in the right way is not trivial, see `On finding the average of two unsigned integers without overflow`

related operations: ceiling division, rounding division

`ai.googleblog.com/2006/06/extra-extra-read-all-about-it-nearly.html`

Potentially Catastrophic Failure

$$51\ days = 51 \cdot 24 \cdot 60 \cdot 60 \cdot 1000 = 4\,406\,400\,000\ ms$$

Boeing 787s must be turned off and on every 51 days to prevent 'misleading data' being shown to pilots

C++ Data Model

Model/Bits	OS	`short`	`int`	`long`	`long long`	`pointer`
ILP32	Windows/Unix 32-b	16	32	32	64	32
LLP64	Windows 64-bit	16	32	32	64	64
LP64	Linux 64-bit	16	32	64	64	64

`char` is always 1 byte

LP32 Windows 16-bit APIs (no more used)

int*_t <cstdint>

C++ provides fixed width integer types.

They have the same size on any architecture:

int8_t, uint8_t

int16_t, uint16_t

int32_t, uint32_t

int64_t, uint64_t

Good practice: Prefer fixed-width integers instead of native types. `int` and `unsigned` can be directly used as they are widely accepted by C++ data models

Fixed Width Integers 2/3

`int*_t` types are not "real" types, they are merely *typedefs* to appropriate fundamental types

C++ standard does not ensure a one-to-one mapping:

- There are **five** distinct *fundamental types* (`char`, `short`, `int`, `long`, `long long`)
- There are **four** `int*_t` *overloads* (`int8_t`, `int16_t`, `int32_t`, and `int64_t`)

ithare.com/c-on-using-int_t-as-overload-and-template-parameters

Warning: I/O Stream interprets `uint8_t` and `int8_t` as `char` and not as integer values

```
int8_t var;
cin >> var;  // read '2'
cout << var; // print '2'
int a = var * 2;
cout << a;   // print '100' !!
```

`size_t` and `ptrdiff_t`

`size_t` `ptrdiff_t` `<cstddef>`

`size_t` and `ptrdiff_t` are *aliases* data types capable of storing the biggest representable value on the current architecture

- `size_t` is an unsigned integer type (of at least 16-bit)
- `ptrdiff_t` is the signed version of `size_t` commonly used for computing pointer differences
- `size_t` is the return type of `sizeof()` and commonly used to represent size measures
- `size_t` / `ptrdiff_t` are 4 bytes on 32-bit architectures, and 8 bytes on 64-bit architectures
- C++23 adds `uz` / `UZ` literals for `size_t`, and `z` / `Z` for `ptrdiff_t`

Signed/Unsigned Integer Characteristics

`Signed` and `Unsigned` integers use the same hardware for their operations, but they have very <u>different semantic</u>

Basic csoncepts:

Overflow The result of an arithmetic operation exceeds the word length, namely the positive/negative largest values

Wraparound The result of an arithmetic operation is reduced modulo 2^N where N is the number of bits of the word

Signed Integer

- Represent positive, negative, and zero values ($\mathbb{Z}$)
- ✅ Represent the human intuition of numbers
- ⚠ More negative values ($2^{31} - 1$) than positive ($2^{31} - 2$)

  Even multiply, division, and modulo by -1 can fail
- ⚠ *Overflow/underflow semantic* $\rightarrow$ undefined behavior

  Possible behavior: overflow: $(2^{31} - 1) + 1 \rightarrow min$

  underflow: $-2^{31} - 1 \rightarrow max$
- ⚠ Bit-wise operations are implementation-defined

  e.g. signed shift $\rightarrow$ undefined behavior
- *Properties*: commutative, reflexive, not associative (overflow/underflow)

Unsigned Integer

- Represent only *non-negative* values ($\mathbb{N}$)
- Discontinuity in 0, $2^{32} - 1$
- ✅ Wraparound semantic $\rightarrow$ <u>well-defined</u> (modulo 2^{32})
- ✅ Bit-wise operations are <u>well-defined</u>
- *Properties*: commutative, reflexive, associative

When Use Signed/Unsigned Integer? 1/2

Google Style Guide

> Because of historical accident, the C++ standard also uses unsigned integers to represent the size of containers - many members of the standards body believe this to be a mistake, but it is effectively impossible to fix at this point

Solution: use `int64_t`

max value: $2^{63} - 1 =$ 9,223,372,036,854,775,807 or
9 quintillion (9 billion of billion),
about 292 years in nanoseconds,
9 million terabytes

When Use Signed/Unsigned Integer? 2/2

When use `signed` integer?

- if it can be mixed with negative values, e.g. subtracting byte sizes
- prefer expressing non-negative values with signed integer and assertions
- optimization purposes, e.g. exploit undefined behavior in loops

When use `unsigned` integer?

- if the quantity can never be mixed with negative values (?)
- bitmask values
- optimization purposes, e.g. division, modulo
- safety-critical system, signed integer overflow could be "non-deterministic"

Subscripts and sizes should be signed, *Bjarne Stroustrup*
Don't add to the signed/unsigned mess, *Bjarne Stroustrup*
Integer Type Selection in C++: in Safe, Secure and Correct Code, *Robert C. Seacord*

Arithmetic Type Limits

Query properties of arithmetic types in C++11:

```
#include <limits>

std::numeric_limits<int>::max();      // 2^31 - 1
std::numeric_limits<uint16_t>::max(); // 65,535

std::numeric_limits<int>::min();      // -2^31
std::numeric_limits<unsigned>::min(); // 0
```

* this syntax will be explained in the next lectures

Promotion and Truncation

Promotion to a larger type keeps the sign

```
int16_t x = -1;
int     y = x; // sign extend
cout << y;     // print -1
```

Truncation to a smaller type is implemented as a modulo operation with respect to the number of bits of the smaller type

```
int     x = 65537; // 2^16 + 1
int16_t y = x;     // x % 2^16
cout << y;         // print 1

int     z = 32769; // 2^15 + 1 (does not fit in a int16_t)
int16_t w = z;     // (int16_t) (x % 2^16 = 32769)
cout << w;         // print -32767
```

```
unsigned a = 10;  // array is small
int      b = -1;
array[10ull + a * b] = 0; // ?
```

☠ Segmentation fault!

```
int f(int a, unsigned b, int* array) {  // array is small
    if (a > b)
        return array[a - b]; // ?
    return 0;
}
```

☠ Segmentation fault for `a < 0`!

```
// v.size() return unsigned
for (size_t i = 0; i < v.size() - 1; i++)
    array[i] = 3; // ?
```

☠ Segmentation fault for `v.size() == 0`!

Easy case:

```
unsigned x = 32;     // x can be also a pointer
x         += 2u - 4; // 2u - 4 = 2 + (2^32 - 4)
                     //        = 2^32 - 2
                     // (32 + (2^32 - 2)) % 2^32
cout << x;           // print 30 (as expected)
```

What about the following code?

```
uint64_t x = 32;     // x can be also a pointer
x         += 2u - 4;
cout << x;
```

More negative values than positive

```
int x = std::numeric_limits<int>::max() * -1; //  (2^31 -1) * -1
cout << x;                                    //  -2^31 +1 ok

int y = std::numeric_limits<int>::min() * -1; // -2^31 * -1
cout << y; // hard to see in complex examples // 2^31 overflow!!
```

A pratical example:

```
void f(int* ptr, int pos) {
    pos++;
    if (pos < 0)
        return;      // <-- the compiler assumes that
    ptr[pos] = 0;    //     signed overflow never happen
}                    //     and removes the if statement
int main() {         // compiled with optimizations
    int tmp[10];     // leads to segmentation faults
    f(tmp, INT_MAX);
}
```

Initialize an integer with a value larger then its range is undefined behavior

```
int z = 3000000000; // undefined behavior!!
```

Bitwise operations on signed integer types is undefined behavior

```
int y = 1 << 12;    // undefined behavior!!
```

Shift larger than #bits of the data type is undefined behavior even for **unsigned**

```
unsigned y = 1u << 32u; // undefined behavior!!
```

Undefined behavior in implicit conversion

```
uint16_t a = 65535; // 0xFFFF
uint16_t b = 65535; // 0xFFFF                 expected: 4'294'836'225
cout << (a * b);    // print '-131071' undefined behavior!! (int overflow)
```

Undefined Behavior

Even worse example:

```
#include <iostream>

int main() {
    for (int i = 0; i < 4; ++i)
        std::cout << i * 1000000000 << std::endl;
}
// with optimizations, it is an infinite loop
// --> 1000000000 * i > INT_MAX
// undefined behavior!!

// the compiler translates the multiplication constant into an addition
```

Why does this loop produce undefined behavior?

Is the following loop safe?

```
void f(int size) {
    for (int i = 1; i < size; i += 2)
        ...
}
```

- What happens if `size` is equal to `INT_MAX`?
- How to make the previous loop safe?
- `i >= 0 && i < size` is not the solution because of *undefined behavior* of signed overflow
- Can we generalize the solution when the increment is `i += step`?

Overflow / Underflow

Detecting wraparound for unsigned integral types is **not trivial**

```
// some examples
bool is_add_overflow(unsigned a, unsigned b) {
    return (a + b) < a || (a + b) < b;
}

bool is_mul_overflow(unsigned a, unsigned b) {
    unsigned x = a * b;
    return a != 0 && (x / a) != b;
}
```

Detecting overflow/underflow for signed integral types is even harder and must be checked before performing the operation

Floating-point Types and Arithmetic

IEEE Floating-Point Standard

IEEE754 is the technical standard for floating-point arithmetic

The standard defines the binary format, operations behavior, rounding rules, exception handling, etc.

First Release	: 1985
Second Release	: 2008. Add 16-bit, 128-bit, 256-bit floating-point types
Third Release	: 2019. Specify `min/max` behavior

see `The IEEE Standard 754: One for the History Books`

IEEE764 technical document:

`754-2019 - IEEE Standard for Floating-Point Arithmetic`

In general, **C/C++ adopts IEEE754 floating-point standard**:

`en.cppreference.com/w/cpp/types/numeric_limits/is_iec559`

32/64-bit Floating-Point

- **IEEE764 Single-precision** (32-bit) `float`

Sign	**Exponent** (or base)	**Mantissa** (or significant)
`1-bit`	`8-bit`	`23-bit`

- **IEEE764 Double-precision** (64-bit) `double`

Sign	**Exponent** (or base)	**Mantissa** (or significant)
`1-bit`	`11-bit`	`52-bit`

128/256-bit Floating-Point

- **IEEE764 Quad-Precision** (128-bit) `std::float128` C++23

Sign	Exponent (or base)	Mantissa (or significant)
`1-bit`	`15-bit`	`112-bit`

- **IEEE764 Octuple-Precision** (256-bit) (not standardized in C++)

Sign	Exponent (or base)	Mantissa (or significant)
`1-bit`	`19-bit`	`236-bit`

16-bit Floating-Point

- **IEEE754 16-bit Floating-point** (`std::binary16`) C++23 → GPU, Arm7

Sign	Exponent	Mantissa
`1-bit`	`5-bit`	`10-bit`

- **Google 16-bit Floating-point** (`std::bfloat16`) C++23 → TPU, GPU, Arm8

Sign	Exponent	Mantissa
`1-bit`	`8-bit`	`7-bit`

half-precision-arithmetic-fp16-versus-bfloat16

8-bit Floating-Point (Non-Standardized in C++/IEEE)

- `E4M3`

Sign	Exponent	Mantissa
1-bit	4-bit	3-bit

- `E5M2`

Sign	Exponent	Mantissa
1-bit	5-bit	2-bit

- Floating Point Formats for Machine Learning, *IEEE draft*
- FP8 Formats for Deep Learning, *Intel, Nvidia, Arm*

- **TensorFloat-32 (TF32)** Specialized floating-point format for deep learning applications

- **Posit** (John Gustafson, 2017), also called *unum III* (*universal number*), represents floating-point values with *variable-width* of exponent and mantissa.
  It is implemented in experimental platforms

- `NVIDIA Hopper Architecture In-Depth`
- `Beating Floating Point at its Own Game: Posit Arithmetic`
- `Posits, a New Kind of Number, Improves the Math of AI`
- `Comparing posit and IEEE-754 hardware cost`

Other Real Value Representations (Non-standardized in C++/IEEE) 2/2

- **Microscaling Formats (MX)** Specification for low-precision floating-point formats defined by AMD, Arm, Intel, Meta, Microsoft, NVIDIA, and Qualcomm. It includes FP8, FP6, FP4, (MX)INT8

- **Fixed-point** representation has a fixed number of digits after the radix point (decimal point). The gaps between adjacent numbers are always equal. The range of their values is significantly limited compared to floating-point numbers. It is widely used on embedded systems

---

- `OCP Microscaling Formats (MX) Specification`

Floating-point number:

- *Radix* (or base): β
- *Precision* (or digits): p
- *Exponent* (magnitude): e
- *Mantissa*: M

$$n = \underbrace{M}_{p} \times \beta^e \quad \rightarrow \quad \text{IEEE754: } 1.M \times 2^e$$

```
float  f1 = 1.3f;   // 1.3
float  f2 = 1.1e2f; // 1.1 · 10^2
float  f3 = 3.7E4f; // 3.7 · 10^4
float  f4 = .3f;    // 0.3
double d1 = 1.3;    // without "f"
double d2 = 5E3;    // 5 · 10^3
```

Exponent Bias

In IEEE754 floating point numbers, the exponent value is offset from the actual value by the **exponent bias**

- The exponent is stored as an unsigned value suitable for comparison
- Floating point values are lexicographic ordered
- For a single-precision number, the exponent is stored in the range $[1, 254]$ (0 and 255 have special meanings), and is biased by subtracting 127 to get an exponent value in the range $[-126, +127]$

0	10000111	11000000000000000000000
$+$	$2^{(135-127)} = 2^8$	$\frac{1}{2^1} + \frac{1}{2^2} = 0.5 + 0.25 = 0.75 \xrightarrow{normal} 1.75$

$$+1.75 * 2^8 = 448.0$$

Floating-point - Normal/Denormal

Normal number

A **normal** number is a floating point value that can be represented with *at least one bit set in the exponent* or the mantissa has all 0s

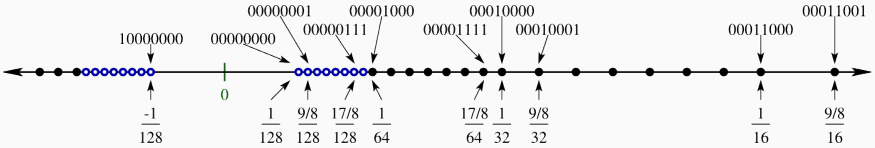
Denormal number

Denormal (or subnormal) numbers fill the underflow gap around zero in floating-point arithmetic. Any non-zero number with magnitude smaller than the smallest normal number is denormal

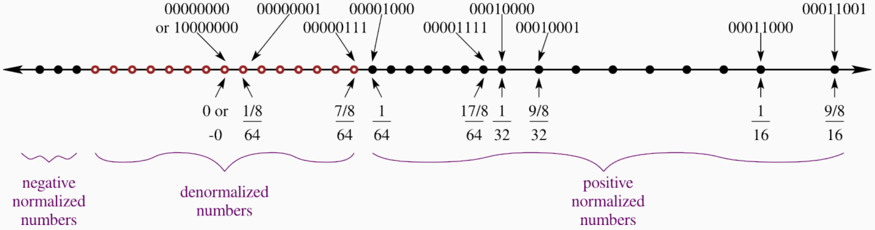
A **denormal** number is a floating point value that can be represented with *all 0s in the exponent*, but the mantissa is non-zero

Floating-point - Normal/Denormal 2/2

Why denormal numbers make sense: (↓ normal numbers)



The problem: distance values from zero (↓ denormal numbers)



Infinity

In the IEEE754 standard, **`inf`** (infinity value) is a numeric data type value that exceeds the maximum (or minimum) representable value

Operations generating **`inf`**:

- $\pm\infty \cdot \pm\infty$
- $\pm\infty \cdot \pm\texttt{finite\_value}$
- $\texttt{finite\_value}\ \mathit{op}\ \texttt{finite\_value} > \texttt{max\_value}$
- $\texttt{non-NaN} \,/\, \pm 0$

There is a single representation for `+inf` and `-inf`

Comparison: $(\texttt{inf == finite\_value}) \rightarrow \texttt{false}$

$(\pm\texttt{inf == }\pm\texttt{inf}) \rightarrow \texttt{true}$

```cpp
cout << 0 / 0;          // undefined behavior
cout << 0.0 / 0.0;      // print "nan"
cout << 5.0 / 0.0;      // print "inf"
cout << -5.0 / 0.0;     // print "-inf"

auto inf = std::numeric_limits<float>::infinity;
cout << (-0.0 == 0.0);                      // true, 0 == 0
cout << ((5.0f / inf) == ((-5.0f / inf));   // true, 0 == 0
cout << (10e40f) == (10e40f + 9999999.0f);  // true, inf == inf
cout << (10e40)  == (10e40f + 9999999.0f);  // false, 10e40 != inf
```

Not a Number (NaN)

NaN

In the IEEE754 standard, `NaN` (not a number) is a numeric data type value representing an undefined or unrepresentable value

Operations generating `NaN`:

- Operations with a `NaN` as at least one operand
- $\pm\infty \cdot \mp\infty$, $0 \cdot \infty$
- $0/0, \infty/\infty$
- $\sqrt{x}$, $\log(x)$ for $x < 0$
- $\sin^{-1}(x), \cos^{-1}(x)$ for $x < -1$ *or* $x > 1$

There are many representations for `NaN` (e.g. $2^{24} - 2$ for `float`)

Comparison: `(NaN == x)` $\rightarrow$ `false`, for every `x`

`(NaN == NaN)` $\rightarrow$ `false`

Machine Epsilon

Machine epsilon

Machine epsilon ε (or *machine accuracy*) is defined to be the smallest number that can be added to 1.0 to give a number other than one

IEEE 754 Single precision : $\varepsilon = 2^{-23} \approx 1.19209 * 10^{-7}$

IEEE 754 Double precision : $\varepsilon = 2^{-52} \approx 2.22045 * 10^{-16}$

Units at the Last Place (ULP)

ULP

Units at the Last Place is the gap between consecutive floating-point numbers

$$ULP(p, e) = \beta^{e-(p-1)} \rightarrow 2^{e-(p-1)}$$

Example:

$$\beta = 10, \ p = 3$$
$$\pi = 3.1415926... \rightarrow x = 3.14 \times 10^0$$
$$ULP(3, 0) = 10^{-2} = 0.01$$

Relation with ε:

- $\varepsilon = ULP(p, 0)$
- $ULP_x = \varepsilon * \beta^{e(x)}$

Floating-Point Representation of a Real Number

The machine floating-point representation $\mathbf{fl}(x)$ of a *real number* x is expressed as

$fl\,(x) = x\,(1+\delta)$, where δ is a small constant

The approximation of a *real number* x has the following properties:

Absolute Error: $|fl(x) - x| \leq \frac{1}{2} \cdot ULP_x$

Relative Error: $\left|\frac{fl\,(x) - x}{x}\right| \leq \frac{1}{2} \cdot \varepsilon$

- NaN (mantissa $\neq 0$)

*	11111111	***********************

- $\pm$ infinity

*	11111111	00000000000000000000000

- Lowest/Largest ($\pm 3.40282 * 10^{+38}$)

*	11111110	11111111111111111111111

- Minimum (normal) ($\pm 1.17549 * 10^{-38}$)

*	00000001	00000000000000000000000

- Denormal number ($< 2^{-126}$)(minimum: $1.4 * 10^{-45}$)

*	00000000	***********************

- ± 0

*	00000000	00000000000000000000000

	`E4M3`	`E5M2`	`half`
Exponent	4 [0*-14] (no inf)	5-bit [0*-30]	
Bias	7	15	
Mantissa	4-bit	2-bit	10-bit
Largest ($\pm$)	$1.75 * 2^8$ 448	$1.75 * 2^{15}$ $57,344$	2^{16} $65,536$
Smallest ($\pm$)	2^{-6} 0.015625	2^{-14} 0.00006	
Smallest (denormal*)	2^{-9} 0.001953125	2^{-16} $1.5258 * 10^{-5}$	2^{-24} $6.0 \cdot 10^{-8}$
Epsilon	2^{-4} 0.0625	2^{-2} 0.25	2^{-10} 0.00098

	`bfloat16`	`float`	`double`
Exponent	8-bit [0*-254]		11-bit [0*-2046]
Bias	127		1023
Mantissa	7-bit	23-bit	52-bit
Largest ($\pm$)	2^{128} $3.4 \cdot 10^{38}$		2^{1024} $1.8 \cdot 10^{308}$
Smallest ($\pm$)	2^{-126} $1.2 \cdot 10^{-38}$		2^{-1022} $2.2 \cdot 10^{-308}$
Smallest (denormal*)	/	2^{-149} $1.4 \cdot 10^{-45}$	2^{-1074} $4.9 \cdot 10^{-324}$
Epsilon	2^{-7} 0.0078	2^{-23} $1.2 \cdot 10^{-7}$	2^{-52} $2.2 \cdot 10^{-16}$

Floating-point - Limits

```
#include <limits>
// T: float or double

std::numeric_limits<T>::max();        // largest value

std::numeric_limits<T>::lowest();     // lowest value (C++11)

std::numeric_limits<T>::min();        // smallest value

std::numeric_limits<T>::denorm_min() // smallest (denormal) value

std::numeric_limits<T>::epsilon();    // epsilon value

std::numeric_limits<T>::infinity()    // infinity

std::numeric_limits<T>::quiet_NaN()  // NaN
```

Floating-point - Useful Functions

```
#include <cmath> // C++11

bool std::isnan(T value)      // check if value is NaN
bool std::isinf(T value)      // check if value is ±infinity
bool std::isfinite(T value)   // check if value is not NaN
                              // and not ±infinity

bool std::isnormal(T value);  // check if value is Normal

T    std::ldexp(T x, p)       // exponent shift x * 2^p
int  std::ilogb(T value)      // extracts the exponent of value
```

Floating-point operations are written

- $\oplus$ addition
- $\ominus$ subtraction
- $\otimes$ multiplication
- $\oslash$ division

$\odot \in \{\oplus, \ominus, \otimes, \oslash\}$

op $\in \{+, -, *, \backslash\}$ denotes exact precision operations

(P1) In general, $a \; op \; b \neq a \odot b$

(P2) **Not Reflexive** $a \neq a$

- *Reflexive* without `NaN`

(P3) **Not Commutative** $a \odot b \neq b \odot a$

- *Commutative* without `NaN` ($\texttt{NaN} \neq \texttt{NaN}$)

(P4) In general, **Not Associative** $(a \odot b) \odot c \neq a \odot (b \odot c)$

(P5) In general, **Not Distributive** $(a \oplus b) \otimes c \neq (a \cdot c) \oplus (b \cdot c)$

(P6) **Identity on operations is not ensured** $(k \oslash a) \otimes a \neq k$

(P7) **No overflow/underflow** Floating-point has *"saturation"* values `inf`, `-inf`

- Adding (or subtracting) can "saturate" before `inf`, `-inf`

C++11 allows determining if a floating-point exceptional condition has occurred by using floating-point exception facilities provided in `<cfenv>`

```
#include <cfenv>
// MACRO
FE_DIVBYZERO  // division by zero
FE_INEXACT    // rounding error
FE_INVALID    // invalid operation, i.e. NaN
FE_OVERFLOW   // overflow (reach saturation value +inf)
FE_UNDERFLOW  // underflow  (reach saturation value -inf)
FE_ALL_EXCEPT // all exceptions

// functions
std::feclearexcept(FE_ALL_EXCEPT); // clear exception status
std::fetestexcept(<macro>);        // returns a value != 0 if an
                                   // exception has been detected
```

Detect Floating-point Errors ★


```cpp
#include <cfenv>    // floating point exceptions
#include <iostream>
#pragma STDC FENV_ACCESS ON // tell the compiler to manipulate the floating-point
                            // environment (not supported by all compilers)
                            // gcc: yes, clang: no
int main() {
    std::feclearexcept(FE_ALL_EXCEPT); // clear
    auto x = 1.0 / 0.0;                // all compilers
    std::cout << (bool) std::fetestexcept(FE_DIVBYZERO); // print true

    std::feclearexcept(FE_ALL_EXCEPT); // clear
    auto x2 = 0.0 / 0.0;               // all compilers
    std::cout << (bool)  std::fetestexcept(FE_INVALID);  // print true

    std::feclearexcept(FE_ALL_EXCEPT); // clear
    auto x4 = 1e38f * 10;              // gcc: ok
    std::cout << std::fetestexcept(FE_OVERFLOW);         // print true
}
```

see What is the difference between quiet NaN and signaling NaN?

Floating-point Issues

Ariene 5: data conversion from 64-bit floating point value to 16-bit signed integer $\rightarrow$ *\$137 million*

Patriot Missile: small chopping error at each operation, 100 hours activity $\rightarrow$ *28 deaths*

Integer type is more accurate than floating type for large numbers

```
cout << 16777217;           // print 16777217
cout << (int) 16777217.0f; // print 16777216!!
cout << (int) 16777217.0;  // print 16777217, double ok
```

`float` numbers are different from `double` numbers

```
cout << (1.1 != 1.1f); // print true !!!
```

The floating point precision is finite!

```
cout << setprecision(20);
cout << 3.33333333f; // print 3.333333254!!
cout << 3.33333333;  // print 3.333333333
cout << (0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1); // print 0.59999999999999998
```

Floating point arithmetic is not associative

```
cout << 0.1 + (0.2 + 0.3) == (0.1 + 0.2) + 0.3; // print false
```

IEEE764 Floating-point computation guarantees to produce **deterministic** output, namely the exact bitwise value for each run, if and only if the **order of the operations is always the same**

$\rightarrow$ *same result on any machine and for all runs*

"Using a double-precision floating-point value, we can represent easily the number of atoms in the universe.

If your software ever produces a number so large that it will not fit in a double-precision floating-point value, chances are good that you have a bug"

***Daniel Lemire**, Prof. at the University of Quebec*

" NASA uses just 15 digits of π to calculate interplanetary travel.
With 40 digits, you could calculate the circumference of a circle the size of the visible universe with an accuracy that would fall by less than the diameter of a single hydrogen atom"

***Latest in space**, Twitter*

Number of atoms in the universe versus floating-point values

Floating-point Algorithms

- `addition` **algorithm** (simplified):

(1) Compare the exponents of the two numbers. Shift the smaller number to the right until its exponent would match the larger exponent
(2) Add the mantissa
(3) Normalize the sum if needed (shift right/left the exponent by 1)

- `multiplication` **algorithm** (simplified):

(1) Multiplication of mantissas. The number of bits of the result is twice the size of the operands ($46 + 2$ bits, with $+2$ for implicit normalization)
(2) Normalize the product if needed (shift right/left the exponent by 1)
(3) Addition of the exponents

- `fused multiply-add` (`fma`):
  - Recent architectures (also GPUs) provide `fma` to compute addition and multiplication in a single instruction (performed by the compiler in most cases)
  - The rounding error of $fma(x, y, z)$ is less than $(x \otimes y) \oplus z$

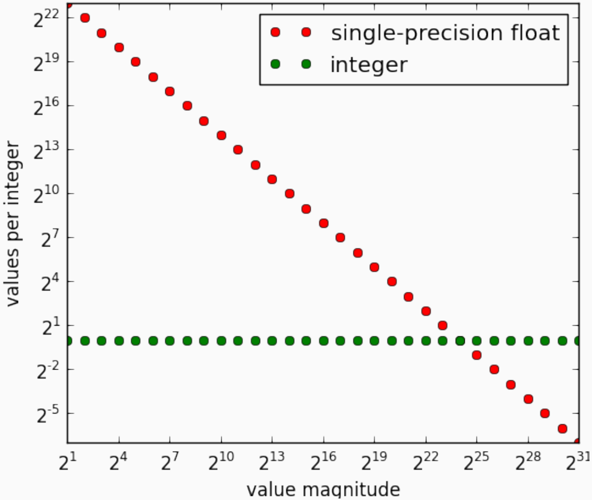
Catastrophic Cancellation

Catastrophic cancellation (or *loss of significance*) refers to loss of relevant information in a floating-point computation that cannot be revered

Two cases:

(C1) $\mathbf{a} \pm \mathbf{b}$, where $\mathbf{a} \gg \mathbf{b}$ or $\mathbf{b} \gg \mathbf{a}$. The value (or part of the value) of the smaller number is lost

(C2) $\mathbf{a} - \mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ are approximation of exact values and $\mathbf{a} \approx \mathbf{b}$, namely a loss of precision in both $\mathbf{a}$ and $\mathbf{b}$. $\mathbf{a} - \mathbf{b}$ cancels most of the relevant part of the result because $\mathbf{a} \approx \mathbf{b}$. It implies a *small absolute error* but a *large relative error*



Intersection $= 16,777,216 = 2^{24}$

Catastrophic Cancellation (case 1) 3/5

How many iterations performs the following code?

```
while (x > 0)
    x = x - y;
```

How many iterations?

```
float:  x = 10,000,000   y = 1     -> 10,000,000
float:  x = 30,000,000   y = 1     -> does not terminate
float:  x =    200,000   y = 0.001 -> does not terminate
bfloat: x =        256   y = 1     -> does not terminate !!
```

Floating-point increment

```
float x = 0.0f;
for (int i = 0; i < 20000000; i++)
x += 1.0f;
```

What is the value of `x` at the end of the loop?

Ceiling division $\left\lceil \frac{a}{b} \right\rceil$

```
//          std::ceil((float) 101 / 2.0f) -> 50.5f -> 51
float x = std::ceil((float) 20000001 / 2.0f);
```

What is the value of `x`?

Catastrophic Cancellation (case 2)

Let's solve a quadratic equation:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$x^2 + 5000x + 0.25$

```
(-5000 + std::sqrt(5000.0f * 5000.0f - 4.0f * 1.0f * 0.25f)) / 2 // x2
(-5000 + std::sqrt(25000000.0f - 1.0f)) / 2 // catastrophic cancellation (C1)
(-5000 + std::sqrt(25000000.0f)) / 2
(-5000 + 5000) / 2 = 0                      // catastrophic cancellation (C2)
// correct result: 0.00005!!
```

relative error: $\frac{|0 - 0.00005|}{0.00005} = 100\%$

The problem

```
cout << (0.11f + 0.11f < 0.22f); // print true!!
cout << (0.1f + 0.1f > 0.2f);    // print true!!
```

Do not use absolute error margins!!

```
bool areFloatNearlyEqual(float a, float b) {
    if (std::abs(a - b) < epsilon); // epsilon is fixed by the user
        return true;
    return false;
}
```

Problems:

- Fixed `epsilon` "looks small" but it could be too large when the numbers being compared are very small
- If the compared numbers are very large, the `epsilon` could end up being smaller than the smallest rounding error, so that the comparison always returns `false`

Solution: Use relative error $\frac{|a-b|}{b} < \varepsilon$

```
bool areFloatNearlyEqual(float a, float b) {
    if (std::abs(a - b) / b < epsilon); // epsilon is fixed
        return true;
    return false;
}
```

Problems:

- `a=0, b=0` The division is evaluated as `0.0/0.0` and the whole `if` statement is `(nan < espilon)` which always returns `false`
- `b=0` The division is evaluated as `abs(a)/0.0` and the whole `if` statement is `(+inf < espilon)` which always returns `false`
- `a and b very small.` The result should be `true` but the division by `b` may produces wrong results
- `It is not commutative.` We always divide by `b`

Possible solution: $\frac{|a-b|}{\max(|a|,|b|)} < \varepsilon$

```
bool areFloatNearlyEqual(float a, float b) {
    constexpr float normal_min      = std::numeric_limits<float>::min();
    constexpr float relative_error = <user_defined>

    if (!std::isfinite(a) || !isfinite(b)) // a = ±∞, NaN or b = ±∞, NaN
        return false;
    float diff  = std::abs(a - b);
    // if "a" and "b" are near to zero, the relative error is less effective
    if (diff <= normal_min) // or also: user_epsilon * normal_min
        return true;

    float abs_a = std::abs(a);
    float abs_b = std::abs(b);
    return (diff / std::max(abs_a, abs_b)) <= relative_error;
}
```

Minimize Error Propagation - Summary

- Prefer **multiplication/division** rather than addition/subtraction
- Try to reorganize the computation to **keep near** numbers with the same scale (e.g. sorting numbers)
- Consider to **put a zero** very small number (under a threshold). Common application: iterative algorithms
- Scale by a **power of two** is safe
- **Switch to log scale**. Multiplication becomes Add, and Division becomes Subtraction
- Use a **compensation algorithm** like Kahan summation, Dekker's FastTwoSum, Rump's AccSum

References

Suggest readings:

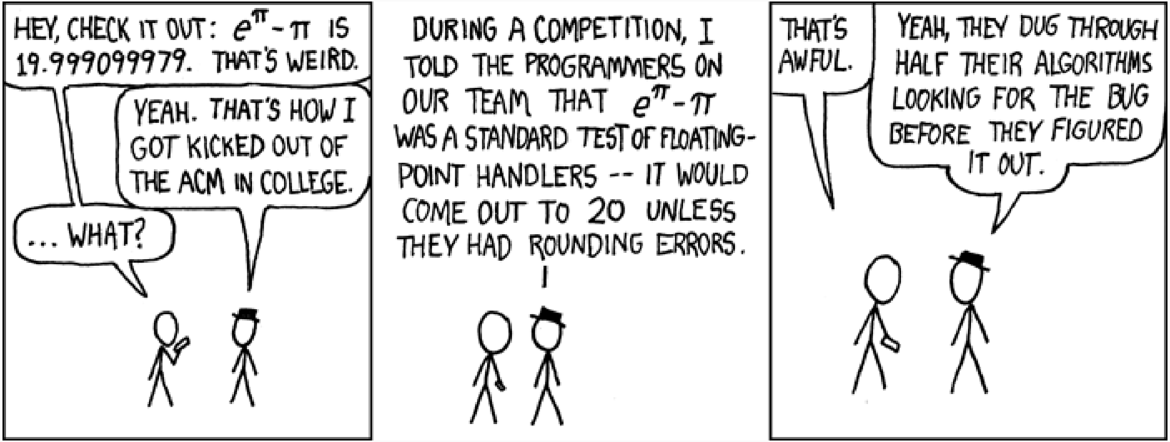
- What Every Computer Scientist Should Know About Floating-Point Arithmetic
- Do Developers Understand IEEE Floating Point?
- Yet another floating point tutorial
- Unavoidable Errors in Computing

Floating-point Comparison readings:

- The Floating-Point Guide - Comparison
- Comparing Floating Point Numbers, 2012 Edition
- Some comments on approximately equal FP comparisons
- Comparing Floating-Point Numbers Is Tricky

Floating point tools:

- IEEE754 visualization/converter
- Find and fix floating-point problems


HEY, CHECK IT OUT: $e^{\pi} - \pi$ IS 19.999099979. THAT'S WEIRD.
YEAH. THAT'S HOW I GOT KICKED OUT OF THE ACM IN COLLEGE.
... WHAT?
DURING A COMPETITION, I TOLD THE PROGRAMMERS ON OUR TEAM THAT $e^{\pi} - \pi$ WAS A STANDARD TEST OF FLOATING-POINT HANDLERS -- IT WOULD COME OUT TO 20 UNLESS THEY HAD ROUNDING ERRORS.
THAT'S AWFUL.
YEAH, THEY DUG THROUGH HALF THEIR ALGORITHMS LOOKING FOR THE BUG BEFORE THEY FIGURED IT OUT.