Modern C++ Programming

2. Basic Concepts I

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Preparation

What Compiler Should I Use?

Popular (free) compilers:

- Microsoft Visual Code (**MSVC**) is the compiler offered by Microsoft
- The GNU Compiler Collection (**GCC**) contains the most popular C++ Linux compiler
- **Clang** is a C++ compiler based on LLVM Infrastructure available for Linux/Windows/Apple (default) platforms

Suggested compiler: **Clang**

- Comparable performance with GCC/MSVC and low memory usage [`compilers comparison link`]
- Expressive diagnostics (examples and propose corrections)
- Strict C++ compliance. GCC/MSVC compatibility (inverse direction is not ensured)
- Includes very useful tools: memory sanitizer, static code analyzer, automatic formatting, linter, etc.
- Easy to install: `releases.llvm.org`

C++ compiler features support: `en.cppreference.com/w/cpp/compiler_support`

Install the Compiler

Install the last gcc/g++ (v9)

```
$ sudo add-apt-repository ppa:ubuntu-toolchain-r/test
$ sudo apt update
$ sudo apt install gcc-9 g++-9
$ gcc-9 --version
```

Install the last clang/clang++ (v9)

```
$ wget https://releases.llvm.org/9.0.0/clang+llvm-9.0.0-x86_64\
-linux-gnu-ubuntu-18.04.tar.xz
$ tar xf clang+llvm-9.0.0-x86_64-linux-gnu-ubuntu-18.04.tar.xz
$ PATH=$PATH:$(pwd)/bin
$ LD_LIBRARY_PATH=$LD_LIBRARY_PATH:$(pwd)/lib64
$ clang-9.0 --version
```

What Editor/IDE Compiler Should I Use?

Popular C++ IDE (Integrated Development Environment) and editors:

- **Microsoft Visual Studio**. (free, Windows)
- **Clion** (link). (free for student). Powerful IDE with a lot of options
- **Atom** (link). Standalone editor oriented for programming (developed by GitHub)
- **Sublime Text editor** (link). Stand-alone editor oriented to programming
- **QT-Creator** (link). Fast (written in C++), simple
- **XCode**, **Eclipse** (**Cevelop**, `www.cevelop.com`), **Vim**, etc.

Not suggested:

- `Notepad, Gedit, and other similar editors`
  Lack of support for programming

How to Compile?

Compile C++11, C++14, C++17 programs:

```
g++ -std=c++11 <program.cpp> -o program
g++ -std=c++14 <program.cpp> -o program
g++ -std=c++17 <program.cpp> -o program
```

Compiler version and C++ Standard:

Compiler	C++11		C++14		C++17	
	Core	Library	Core	Library	Core	Library
g++	4.8.1	5.1	5.1	5.1	7.1	ongoing
clang++	3.3	3.3	3.4	3.5	5.0	ongoing
MSVC	19.0	19.0	19.10	19.0	19.14	19.14+

en.cppreference.com/w/cpp/compiler_support

Hello World

C code with `printf`:

```c
#include <stdio.h>

int main() {
    printf("Hello World!\n");
}
```

`printf` prints on standard output

C++ code with streams:

```cpp
#include <iostream>

int main() {
    std::cout << "Hello World!\n";
}
```

`cout` : represent the standard output stream

The previous example can be written with the global `std` namespace:

```cpp
#include <iostream>
using namespace std;

int main() {
    cout << "Hello World!\n";
}
```

`std::cout` is an example of *output* stream. Data is redirected to a destination, in this case the destination is the standard output

C:

```c
#include <stdio.h>

int main() {
    int    a = 4;
    double b = 3.0;
    char*  c = "hello";
    printf("%d %f %s\n", a, b, c);
}
```

C++:

```cpp
#include <iostream>

int main() {
    int    a = 4;
    double b = 3.0;
    char*  c = "hello";
    std::cout << a << " " << b << " " << c << "\n";
}
```

I/O Stream (Why should we prefer I/O stream?) 2/3

- **Type-safe**: The type of object pass to I/O stream is known statically by the compiler. In contrast, `printf` uses "%" fields to figure out the types dynamically
- **Less error prone**: With IO Stream, there are no redundant "%" tokens that have to be consistent with the actual objects pass to I/O stream. Removing redundancy removes a class of errors
- **Extensible**: The C++ IO Stream mechanism allows new user-defined types to be pass to I/O stream without breaking existing code
- **Comparable performance**: If used correctly may be faster than C I/O (`printf`, `scanf`, `etc`)

- Forget the number of parameters:

```
printf("long phrase %d long phrase %d", 3);
```

- Use the wrong format:

```
int a = 3;
...many lines of code...
printf(" %f", a);
```

- The "%c" conversion specifier does not automatically skip any leading white space:

```
scanf("%d", &var1);
scanf(" %c", &var2);
```

C++ Primitive Types

Type	Size (bytes)	Range	Fixed width types
`bool`	1	true, false	
`char` †	1	-127 to 127	
`signed char`	1	-128 to 127	int8_t
`unsigned char`	1	0 to 255	uint8_t
`short`	2	-2^{15} to $2^{15}-1$	int16_t
`unsigned short`	2	0 to $2^{16}-1$	uint16_t
`int`	4	-2^{31} to $2^{31}-1$	int32_t
`unsigned int`	4	0 to $2^{32}-1$	uint32_t
`long int`	4/8*		int32_t/int64_t
`long unsigned int`	4/8*		uint32_t/uint64_t
`long long int`	8	-2^{63} to $2^{63}-1$	int64_t
`long long unsigned int`	8	0 to $2^{64}-1$	uint64_t
`float` (IEEE 754)	4	$\pm 1.18 \times 10^{-38}$ to $\pm 3.4 \times 10^{+38}$	
`double` (IEEE 754)	8	$\pm 2.23 \times 10^{-308}$ to $\pm 1.8 \times 10^{+308}$	

* 4 bytes on Windows64 systems, † one-complement

- **Any other entity in C++ is**
  - an *alias* to the correct type depending to the context and the architectures
  - a *composition* of builtin types: `struct`, `class`, `union`, etc.
- Interesting: C++ does not explicitly define the size of a byte (see `Exotic architectures the standards committees care about`)

Builtin Types - Short Name

Signed Type	short name
`signed char`	/
`signed short int`	`short`
`signed int`	`int`
`signed long int`	`long`
`signed long long int`	`long long`

Unsigned Type	short name
`unsigned char`	/
`unsigned short int`	`unsigned short`
`unsigned int`	`unsigned`
`unsigned long int`	`unsigned long`
`unsigned long long int`	`unsigned long long`

en.cppreference.com/w/cpp/language/types

en.cppreference.com/w/cpp/types/integer

Builtin Types - Suffix and Prefix

Type	SUFFIX	example
`int`	NO PREFIX	2
`unsigned int`	u	3u
`long int`	l	8l
`long unsigned`	ul	2ul
`long long int`	ll	4ll
`long long unsigned int`	ull	7ull
`float`	f	3.0f
`double`		3.0

Representation	PREFIX	example
`Binary` C++14	0b	0b010101
`Octal`	0	0308
`Hexadecimal`	0x or 0X	0xFFA010

Conversion Rules

Implicit type conversion rules (applied in order) :

$\otimes$: any operations (*, +, /, -, %, etc.)

(a) Floating point promotion

`floating_type` $\otimes$ `integer_type` = `floating_type`

(b) Size promotion

`small_type` $\otimes$ `large_type` = `large_type`

(c) Sign promotion

`signed_type` $\otimes$ `unsigned_type` = `unsigned_type`

Common Errors

- Integers are not floating points!

```
int   b = 7;
float a = b / 2;   // a = 3 not 3.5!!
int   a = b / 2.0; // again a = 3 not 3.5!!
```

- Integer type are more accurate than floating types for large numbers!!

```
cout << 16777217;            // print 16777217
cout << (int) 16777217.0f; // print 16777216!!
cout << (int) 16777217.0;  // print 16777217, double ok
```

- `float` numbers are different from `double` numbers!

```
cout << (1.1 != 1.1f); // print true !!!
```

Other Data Types

- C++ provides also `long double` (no IEEE-754) of size 8/12/16 bytes depending on the implementation

- C++ does not provide support for **half float** (16-bit) data type (IEEE 754-2008)
  - Some compilers already provide support for half float (GCC for ARM: `__fp16`, LLVM compiler: `half`)
  - Some modern CPUs (+ Nvidia GPUs) provide half-float instructions
  - There is a proposal (next standard) since 2016
  - Software support (OpenGL, Photoshop, Lightroom, `half.sourceforge.net`)

size_t and std::byte

size_t <cstddef>

`size_t` is a data type (alias) capable of storing the biggest representable value on the current architecture

- `size_t` is an unsigned integer type (of at least 16-bit)
- In common C++ implementations:
  - `size_t` is 4 bytes on 32-bit architectures
  - `size_t` is 8 bytes on 64-bit architectures
- `size_t` is commonly used to represent size measures

C++17 defines also `std::byte` type to represent a collection of bit (`<cstddef>`). It supports only bitwise operations (no conversions or arithmetic operations)

`void` Type

`void` is an incomplete type (not defined) without a values

- `void` indicates also a function has no return type
  e.g. `void f()`
- `void` indicates also a function has no parameters
  e.g. `f(void)`
- In C `sizeof(void) == 1` (GCC), while in C++ `sizeof(void)` does not compile!!

```
int main() {
//  sizeof(void); // compile error!!
}
```

Pointer type

The **type of a pointer** (e.g. `void*`) is an *unsigned* integer of 32-bit/64-bit depending on the underlying architecture

- It only supports the operators `+, -, ++, --` and comparisons `==, !=, <, <=, >, >=`
- A pointer cannot be implicitly converted to an integer type

```
void*  x;
size_t y = (size_t) x; // ok
// size_t y = x;       // compile error
```

nullptr Keyword

C++11 introduces the new keyword `nullptr` to represent `null` pointers (instead of `NULL` macro)

```
int* p1 = NULL;    // ok, equal to int* p1 = 0
int* p2 = nullptr; // ok

int n1 = NULL;     // ok, we are assigning 0 to n1
// int n2 = nullptr;  // error! we are assigning a null pointer
//                       to an integer variable

// int* p2 = true ? 0 : nullptr; // incompatible types
```

Remember: `nullptr` is not a pointer, but an object of type `nullptr_t` → safer

Integral Data Types

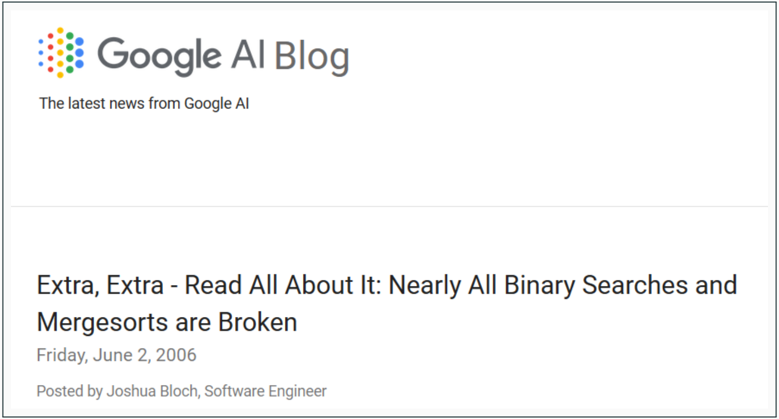
A Firmware Bug

"Certain SSDs have a firmware bug causing them to irrecoverably fail after exactly 32,768 hours of operation. SSDs that were put into service at the same time will fail simultaneously, so RAID won't help"

HPE SAS Solid State Drives - Critical Firmware Upgrade

Via twitter.com/martinkl/status/1202235877520482306?s=19

Overflow Implementations


Google AI Blog

The latest news from Google AI

Extra, Extra - Read All About It: Nearly All Binary Searches and Mergesorts are Broken

Friday, June 2, 2006

Posted by Joshua Bloch, Software Engineer

other examples: average, ceiling division, rounding division

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ai.googleblog.com/2006/06/extra-extra-read-all-about-it-nearly.html

C++ provides fixed width integer types. They have the same size on any architecture (`#include <cstdint>`)

int8_t, uint8_t,

int16_t, uint16_t,

int32_t, uint32_t,

int64_t, uint64_t

Warning: I/O Stream interprets `uint8_t` and `int8_t` as `char` and not as integer values

```
int8_t var;
std::cin >> var;  // read '2'
std::cout << var; // print '2'
int a = var * 2;
std::cout << a;   // print 100 !!
```

Fixed Width Integers

`int*_t` types are <u>not</u> “real” types, they are merely *typedefs* to appropriate fundamental types

C++ standard does not ensure an one-to-one mapping:

- There are **five** distinct *fundamental types* (`char` , `short` , `int` , `long` , `long long`)
- There are **four** `int*_t` *overloads* (`int8_t` , `int16_t` , `int32_t` , and `int64_t`)

```
#include <cstddef>
void f(int8_t x)  {}
void f(int16_t x) {}
void f(int32_t x) {}
void f(int64_t x) {}
int main() {
    int x = 0;
    f(x); // compile error!! under 32-bit ARM GCC
} // "int" is not mapped to int*_t type in this (very) particular case
```

Full Story: `ithare.com/c-on-using-int_t-as-overload-and-template-parameters`

`Signed` and **`unsigned`** integers use the same hardware for their operations, but they have very different semantic:

`signed` integers

- Represent positive, negative, and zero values ($\mathbb{Z}$)
- More negative values ($2^{31} - 1$) than positive ($2^{31} - 2$)
- Overflow/underflow is undefined behavior
  Possible behavior:
  overflow: $2^{31} + 1 \rightarrow -1$
  underflow: $(-2^{31} + 1) - 1 \rightarrow 0$
- Bit-wise operations are implementation-defined
- Commutative, reflexive, not associative (overflow)

`unsigned` integers

- Represent only *non-negative* values ($\mathbb{N}$)
- Overflow/underflow is well-defined (modulo 2^{32})
- Discontinuity in 0, $2^{32} - 1$
- Bit-wise operations are well-defined
- Commutative, reflexive, associative

Google Style Guide

> Because of historical accident, the C++ standard also uses unsigned integers to represent the size of containers - many members of the standards body believe this to be a mistake, but it is effectively impossible to fix at this point

Solution: use `int64_t`

max value: $2^{63} - 1 =$ 9,223,372,036,854,775,807 or
9 quintillion (9 billion of billion),
about 292 years (nanoseconds),
9 million terabytes

Builtin type limits

Query properties of arithmetic types in C++11:

```
#include <limits>

std::numeric_limits<int>::max();            // 2^31 - 1
std::numeric_limits<unsigned short>::max(); // 65,535

std::numeric_limits<int>::min();            // -2^31
std::numeric_limits<unsigned>::min();       // 0
```

Promotion, Truncation, and Shift

Promotion to a larger type keeps the sign

```
int16_t x = -1;
int     y = x; // sign extend
cout << y;     // print 1
auto    z  = 4294967296; // 2^32 ok, z is |\texttt{int64_t}|
// auto z1 = 1 << 32;    // wrong!!, z1 is undefined (int)
```

Truncation to a smaller type is implemented as a modulo operation and keeps the sign

```
int     x = 65537; // 2^16 + 1
int16_t y = x;     // x % 2^16
cout << y;         // print 1
```

Implicit Conversion

Integral data types smaller than 32-bit are *implicitly* promoted to `int`, independently if they are *signed* or *unsigned*

- Unary `+, -, ~` and Binary `+, -, &, etc.` promotion:

```
char a = 48;       // '0'
cout << a;         // print '0'
cout << +a;        // print '48'
cout << (a + 0); // print '48'

unsigned char a1 = 255;
unsigned char b1 = 255;
cout << (a1 + b1);  // print '510' (no overflow)
```

Mixing Signed/Unsigned 1/2

Common errors:

```
unsigned a = 10;
int      b = -1;
array[10ull + a * b] = 0;
```

☠ Segmentation fault!

```
int f(int a, unsigned b, int* array) {
    if (a > b)
        return array[a - b];
    return 0;
}
```

☠ Segmentation fault!

```
// v.size() return unsigned
for (size_t i = 0; i < v.size() - 1; i++)
    array[i] = 3;
```

☠ Segmentation fault for v.size() = 0!

Easy case:

```
unsigned x = 32;      // x can be also a pointer
x         += 2u - 4; // -4 = 2^32 - 2
                     // ((2^32 - 2) + 32) % 2^32
cout << x + 32;      // print 30
```

What about the following code?

```
uint64_t x = 32;
x         += 2u - 4;
cout << x + 32;
```

More negative values than positive

```
int x = std::numeric_limits<int>::max() * -1; //  (2^31 -1) * -1
cout << x;                                    //  -2^31 +1 ok

int y = std::numeric_limits<int>::min() * -1; // -2^31 * -1
cout << y;                                    // 2^31 overflow!!
```

A pratical example:

```
void f(int* ptr, int size) {
    size += 1 ;
    if (size < 0) return; // <-- the compiler assumes that
    ptr[pos] = 0;         // signed overflow never happen
}                         // and removes the if statement
int main() {          // compiled with optimizations
    int tmp[10];      // leads to segmentation faults
    f(tmp, INT_MAX);
}
```

---

twitter.com/shafikyaghmour/status/1134578146781491201

Even worse example:

```
#include <iostream>

int main() {
    for (int i = 0; i < 4; ++i)
        std::cout << i * 1000000000 << std::endl;
}
// with optimizations, it is a infinite loop (undefined behavior)

// the compiler translates the multiplication constant
// into an addition
```

Why does this loop produce undefined behavior?

Shift larger than #bits of the data type is undefined behavior even for **unsigned**

```
unsigned x = 1;
unsigned y = x >> 32; // undefined behavior!!
```

Undefined behavior in implicit conversion

```
unsigned short a2 = 65535; // 0xFFFF
unsigned short b2 = 65535; // 0xFFFF
cout << (a2 * b2);         // print '-131071' (0xFFFE0001)
                           // undefined behavior (int overflow)
```

Overflow / Underflow

Floating point types have infinity values (`+inf`, `-inf`) and no overflow/underflow behavior

Detect overflow/underflow for unsigned integral types is **not trivial**

```
bool isAddOverflow(unsigned a, unsigned b) {
    return (a + b) < a || (a + b) < b;
}

bool isMulOverflow(unsigned a, unsigned b) {
    unsigned x = a * b;
    return a != 0 && (x / a) != b;
}
```

Overflow/underflow for signed integral types is **not defined** !!
Undefined behavior must be checked before perform the operation

Floating-point Arithmetic

32/64-bit Floating-Point

IEEE754 is the technical standard for floating-point arithmetic

The standard defines the binary format, operations behavior, rounding rules, exception handling, etc.

Releases:

- First: 1985
- Second: 2008. Add 16-bit floating point
- Third: 2019. Specify `min/max` behavior

IEEE764 technical document:

`754-2019 - IEEE Standard for Floating-Point Arithmetic`

In general, *C/C++ adopts IEEE754 floating-point standard*

32/64-bit Floating-Point

- **IEEE764 Single precision** (32-bit) (`float`)

Sign	**Exponent** (or base)	**Mantissa** (or significant)
1-bit	8-bit	23-bit

- **IEEE764 Double precision** (64-bit) (`double`)

Sign	**Exponent** (or base)	**Mantissa** (or significant)
1-bit	11-bit	52-bit

16-bit Floating-Point

- **IEEE754 16-bit Floating-point** (`fp16`)

Sign	Exponent	Mantissa
1-bit	5-bit	10-bit

- **Google 16-bit Floating-point** (`bfloat16`)

Sign	Exponent	Mantissa
1-bit	8-bit	7-bit

half-precision-arithmetic-fp16-versus-bfloat16

Other Real Value Representations (non-standard)

- **Posit** (John Gustafson, 2017), also called *unum III* (*universal number*), represents floating-point values with *variable-width* of exponent and mantissa

- **Fixed-point** representation has a fixed number of digits after the radix point (decimal point). The gaps between adjacent numbers are always equal. The range of their values is significantly limited compared to floating-point numbers

---

Reference:

`Beating Floating Point at its Own Game: Posit Arithmetic`

Exponent Bias

In IEEE754 floating point numbers, the exponent value is offset from the actual value by the **exponent bias**

- The exponent is stored as an unsigned value suitable for comparison
- Floating point values are lexicographic ordered
- For a single-precision number, the exponent is stored in the range $[1, 254]$ (0 and 255 have special meanings), and is biased by subtracting 127 to get an exponent value in the range $[-126, +127]$

- Example

0	10000111	11000000000000000000000
+	$2^{(135-127)} = 2^8$	$\frac{1}{2^1} + \frac{1}{2^2} = 0.5 + 0.25 = 0.75 \overset{normal}{\rightarrow} 1.75$

$$+1.75 * 2^8 = 448.0$$

Floating-point number:

- *Radix* (or base): β
- *Precision* (or digits): p
- *Exponent*: e
- *Mantissa*: M

$$n = \underbrace{M}_{p} \times \beta^e \quad \rightarrow \quad \text{IEEE754: } 1.M \times 2^e$$

Some examples:

```
float  f1 = 1.3f;   // 1.3
float  f2 = 1.1e2f; // 1.1 · 10^2
float  f3 = 3.7E4f; // 3.7 · 10^4
float  f4 = .3f;    // 0.3
double d1 = 1.3;    // without "f"
double d2 = 5E3;    // 5 · 10^3
```

Normal number

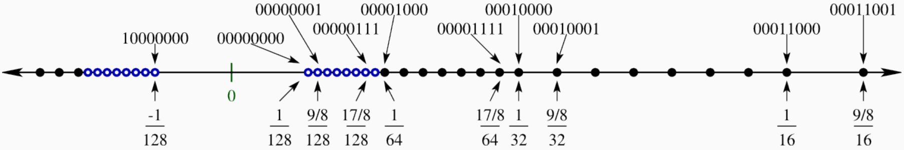
A **normal** number is a floating point number that can be represented *without leading zeros in its mantissa* (one in the first left position) and at least one bit set in the exponent

Denormal number

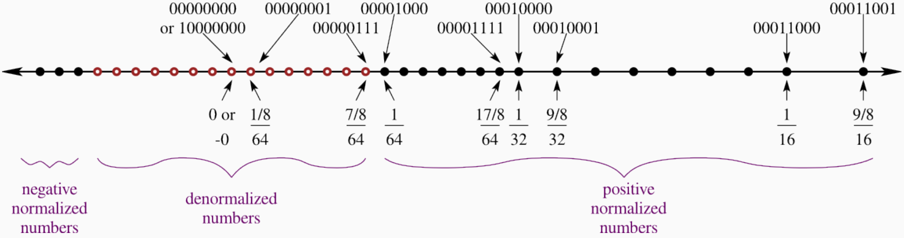
Denormal (or subnormal) numbers fill the underflow gap around zero in floating-point arithmetic. Any non-zero number with magnitude smaller than the smallest normal number is denormal

If *the exponent is all 0s*, but the mantissa is non-zero (else it would be interpreted as zero), then the value is a denormal number

Why denormal numbers make sense: (↓ normal numbers)



The problem: distance values from zero (↓ denormal numbers)



reference: `www.toves.org/books/float/`

Floating-point - Special Values

- ± infinity

*	11111111	00000000000000000000000

- `NaN` (mantissa $\neq 0$)

*	11111111	***********************

- ± 0

*	00000000	00000000000000000000000

- Denormal number ($< 2^{-126}$)(minimum: $1.4 * 10^{-45}$)

*	00000000	***********************

- Minimum (normal) ($\pm 1.17549 * 10^{-38}$)

*	00000001	00000000000000000000000

- Lowest/Largest ($\pm 3.40282 * 10^{+38}$)

*	11111110	11111111111111111111111

Machine Epsilon

Machine epsilon

Machine epsilon ε (or *machine accuracy*) is defined to be the smallest number that can be added to 1.0 to give a number other than one

IEEE 754 Single precision : $\varepsilon = 2^{-23} \approx 1.19209 * 10^{-7}$

IEEE 754 Double precision : $\varepsilon = 2^{-52} \approx 2.22045 * 10^{-16}$

Units at the Last Place

ULP

Units at the Last Place is the gap between consecutive floating-point numbers

$$ULP(p, e) = 1.0 \times \beta^{e-(p-1)}$$

Example:

$$\beta = 10, \; p = 3$$
$$\pi = 3.1415926... \rightarrow x = 3.14 \times 10^0$$
$$ULP(3, 0) = 10^{-2} = 0.01$$

Relation with ε:

- $\varepsilon = ULP(p, 0)$
- $ULP_x = \varepsilon * \beta^{e(x)}$

Floating-point Error

Machine floating-point representation of x is denoted **fl(**x**)**

$$fl\,(x) = x\,(1+\delta)$$

Absolute Error: $|fl(x) - x| \leq \frac{1}{2} \cdot ULP_x$

Relative Error: $\left|\frac{fl\,(x) - x}{x}\right| \leq \frac{1}{2} \cdot \varepsilon$

Floating-point Summary

	`half`	`bfloat16`	`float`	`double`
exponent	5-bit [0*-30]	8-bit [0*-254]		11-bit [0*-2046]
bias	15	127		1023
mantissa	10-bit	7-bit	23-bit	52-bit
largest ($\pm$)	2^{16} $65,536$	2^{128} $3.4 \cdot 10^{38}$		2^{1024} $1.8 \cdot 10^{308}$
smallest ($\pm$)	2^{-14} 0.00006	2^{-126} $1.2 \cdot 10^{-38}$		2^{-1022} $2.2 \cdot 10^{-308}$
smallest (denormal)	2^{-24} $6.0 \cdot 10^{-8}$	/	2^{-149} $1.4 \cdot 10^{-45}$	2^{-1074} $4.9 \cdot 10^{-324}$
epsilon	2^{-10} 0.00098	2^{-7} 0.0078	2^{-23} $1.2 \cdot 10^{-7}$	2^{-52} $2.2 \cdot 10^{-16}$

Floating-point - C++ limits

`T`: float or double

```
#include <limits>

// Check if the actual C++ implementation adopts
// the IEEE754 standard:
std::numeric_limits<T>::is_iec559;   // should return true

std::numeric_limits<T>::max();       // largest value

std::numeric_limits<T>::lowest();    // lowest value (C++11)

std::numeric_limits<T>::min();       // smallest value

std::numeric_limits<T>::denorm_min() // smallest (denormal) value

std::numeric_limits<T>::epsilon();   // epsilon value
```

NaN

In the IEEE754 standard, **NaN** (not a number) is a numeric data type value representing an undefined or unrepresentable value

Operations generating `NaN`:

- Operations with a NaN as at least one operand
- $\pm\infty \cdot \mp\infty$, $0 \cdot \infty$
- $0/0, \infty/\infty$
- $\sqrt{x} \mid x < 0$
- $\log(x) \mid x < 0$
- $\sin^{-1}(x), \cos^{-1}(x) \mid x < -1 \; or \; x > 1$

There are many representations for NaN (e.g. $2^{24} - 2$ for `float`)

Comparison: `(NaN == x)` $\rightarrow$ `false`, for every `x`

`(NaN == NaN)` $\rightarrow$ `false`

`inf` Properties

inf

In the IEEE754 standard, `inf` (infinity value) is a numeric data type value that exceeds the maximum (or minimum) representable value

Operations generating `inf`:

- $\pm\infty \cdot \pm\infty$
- $\pm\infty \cdot \pm$`finite_value`
- `finite_value` *op* `finite_value` $>$ `max_value`
- `non-NaN` $/ \pm 0$

There is a single representation for `+inf` and `-inf`

Comparison: (`inf == finite_value`) $\rightarrow$ `false`

(`±inf == ±inf`) $\rightarrow$ `true`

Floating-point - Useful Functions

```
#include <limits>

std::numeric_limits<T>::infinity()  // infinity
std::numeric_limits<T>::quiet_NaN() // NaN
```

```
#include <cmath> // C++11

bool std::isnan(T value)      // check if value is NaN
bool std::isinf(T value)      // check if value is ±infinity
bool std::isfinite(T value)   // check if value is not NaN
                              // and not ±infinity

bool std::isnormal(T value); // check if value is normal

T   std::ldexp(T x, p)        // exponent shift x * 2^p
int std::ilogb(T value)       // extracts the exponent of value
```

Floating-point Special Values Behavior

```
// divide by zero behavior
cout << 0 / 0;          // undefined behavior (integer)
cout << 0.0 / 0.0;      // print "nan"
cout << 5.0 / 0.0;      // print "inf"
cout << -5.0 / 0.0;     // print "-inf"

// dive by infinity behavior
auto inf = std::numeric_limits<float>::infinity;
cout << ((5.0f / inf) == ((-5.0f / inf)); // true

// positive/negative zero comparison
cout << (-0.0 == 0.0); // true

// infnity comparison
cout << (10e40f) == (10e40f + 9999999.0f); // true (inf == inf)
cout << (10e40)  == (10e40 + 9999999.0);   // false (double)
```

C++11 allows determining if a floating-point exceptional condition has occurred by using floating-point exception facilities provided in `<cfenv>`

```
#include <cfenv>
// MACRO
FE_DIVBYZERO  // division by zero
FE_INEXACT    // rounding error
FE_INVALID    // invalid operation, i.e. NaN
FE_OVERFLOW   // overflow (reach saturation value +inf)
FE_UNDERFLOW  // underflow  (reach saturation value -inf)
FE_ALL_EXCEPT // all exceptions

// functions
std::feclearexcept(FE_ALL_EXCEPT); // clear exception status
std::fetestexcept(<macro>);        // returns a value != 0 if an
                                   // exception has been detected
```

Detect Floating-point Errors

```cpp
#include <cfenv>    // floating point exceptions
#include <iostream>

#pragma STDC FENV_ACCESS ON // tell the compiler to manipulate
                            // the floating-point environment
                            // (not supported by all compilers)
                            // gcc: yes, clang: no
int main() {
    std::feclearexcept(FE_ALL_EXCEPT); // clear
    auto x = 1.0 / 0.0; // all compilers
    std::cout << (bool) std::fetestexcept(FE_DIVBYZERO); // print true

    std::feclearexcept(FE_ALL_EXCEPT); // clear
    auto x2 = 0.0 / 0.0; // all compilers
    std::cout << (bool)  std::fetestexcept(FE_INVALID);  // print true

    std::feclearexcept(FE_ALL_EXCEPT); // clear
    auto x4 = 1e38f * 10; // gcc: ok
    std::cout << std::fetestexcept(FE_OVERFLOW);         // print true
}
```

Floating-point Arithmetic Properties 1/2

Floating-point operations are written

- $\oplus$ addition
- $\ominus$ subtraction
- $\otimes$ multiplication
- $\oslash$ division

$\odot \in \{\oplus, \ominus, \otimes, \oslash\}$

op $\in \{+, -, *, \backslash\}$ denotes exact precision operations

Floating-point Arithmetic Properties 2/2

(P1) In general, $a \; op \; b \neq a \odot b$

(P2) **Not Reflexive** $a \odot a$

- *Reflexive* without `NaN`

(P3) **Not Commutative** $a \odot b \neq b \odot a$

- *Commutative* without `NaN` (`NaN` $\neq$ `NaN`)

(P4) In general, **Not Associative** $(a \odot b) \odot c \neq a \odot (b \odot c)$

(P5) In general, **Not Distributive** $(a \oplus b) \otimes c \neq (a \cdot c) \oplus (b \cdot c)$

(P6) **Identity on operations is not ensured** $(k \oslash a) \otimes a \neq a$

(P7) **No overflow/underflow** Floating-point has *"saturation"* values `inf`, `-inf`

- Adding (or subtracting) can "saturate" before `inf`, `-inf`

Floating-point Issues

Ariene 5: data conversion from 64-bit floating point value to 16-bit signed integer
→ *$137 million*

Patriot Missile: small chopping error at each operation, 100 hours activity
→ *28 deaths*

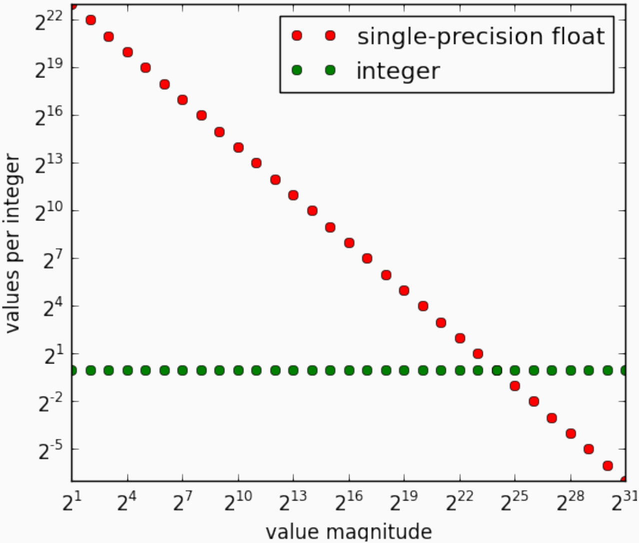
The floating point precision is finite!

```
cout << setprecision(20);
cout << 3.33333333f; // print 3.333333254!!
cout << 3.33333333;  // print 3.333333333
cout << (0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1);
// print 0.59999999999999998
```

Floating point arithmetic is not associative

```
cout << 0.1 + (0.2 + 0.3) == (0.1 + 0.2) + 0.3; // print false
```

IEEE764 Floating-point computation guarantees to produce **deterministic** output, namely the exact bitwise value for each run, if and only if the **order of the operations is always the same**
→ *same result on any machine and for all runs*



Intersection $= 16,777,216 = 2^{24}$

Floating-point increment

```
float x = 0.0f;
for (int i = 0; i < 20000000; i++)
    x += 1.0f;
```

What is the value of `x` at the end of the loop?

Ceiling division $\left\lceil \frac{a}{b} \right\rceil$

```
//         std::ceil((float) 101 / 2.0f) -> 50.5f -> 51
float x = std::ceil((float) 20000001 / 2.0f);
```

The problem

```
cout << (0.11f + 0.11f < 0.22f); // print true!!
cout << (0.1f + 0.1f > 0.2f);    // print true!!
```

Do not use absolute error margins!!

```
bool areFloatNearlyEqual(float a, float b) {
    if (std::abs(a - b) < epsilon); // epsilon is fixed by the user
        return true
    return false;
}
```

Problems:

- Fixed `epsilon` "looks small" but, it could be too large when the numbers being compared are very small
- If the compared numbers are very large, the epsilon could end up being smaller than the smallest rounding error, so that the comparison always returns `false`

Floating-point Comparison 2/3

Solution: Use relative error $\frac{|a-b|}{b} < \varepsilon$

```
bool areFloatNearlyEqual(float a, float b) {
    if (std::abs(a - b) / b < epsilon); // epsilon is fixed
        return true
    return false;
}
```

Problems:

- **`a=0, b=0`** The division is evaluated as `0.0/0.0` and the whole `if` statement is `(nan < espilon)` which always returns `false`
- **`b=0`** The division is evaluated as `abs(a)/0.0` and the whole `if` statement is `(+inf < espilon)` which always returns `false`
- **`a and b very small.`** The result should be `true` but the division by `b` may produces wrong results
- **`It is not commutative.`** We always divide by `b`

Possible solution: $\frac{|a-b|}{\max(|a|,|b|)} < \varepsilon$

```
bool areFloatNearlyEqual(float a, float b) {
    const float normal_min     = std::numeric_limits<float>::min();
    const float relative_error = <user_defined>

    if (std::isfinite(a) || isfinite(b)) // a = ±∞, b = ±∞ and NaN
        return false;
    float diff  = std::abs(a - b);
    // if "a" and "b" are near to zero, the relative error is less
    // effective
    if (diff <= normal_min)
        return true; // or also: user_epsilon * normal_min

    float abs_a = std::abs(a);
    float abs_b = std::abs(b);
    return (diff / std::max(abs_a, abs_b)) <= relative_error;
}
```

Floating-point Algorithms

- `addition` **algorithm** (simplified):

(1) Compare the exponents of the two numbers. Shift the smaller number to the right until its exponent would match the larger exponent

(2) Add the mantissa

(3) Normalize the sum if needed (shift right/left the exponent)

- `multiplication` **algorithm** (simplified):

(1) Multiplication of mantissas. The number of bits of the result is twice the size of the operands (46 + 2 bits, +2 for implicit normalization)

(2) Normalize the product if needed (shift right/left the exponent)

(3) Addition of the exponents

- `fused multiply-add` (`fma`):
    - Recent architectures (also GPUs) provide `fma` to compute these two operations in a single instruction (performed by the compiler)
    - The rounding error is lower $fl(fma(x, y, z)) < fl((x \otimes y) \oplus z)$

Catastrophic Cancellation

Catastrophic cancellation (or *loss of significance*) refers to loss of relevant information in a floating-point computation that cannot be revered

Two cases:

(1) $\mathbf{a} \pm \mathbf{b}$, where $\mathbf{a} \gg \mathbf{b}$ or $\mathbf{b} \gg \mathbf{a}$. The value (or part of the value) of the smaller number is lost

(2) $\mathbf{a} - \mathbf{b}$, where $\mathbf{a} \approx \mathbf{b}$. Loss of precision in both a and b. It implies large relative error

How many iterations performs the following code?

```
while (x > 0)
    x = x - y;
```

```
float:  x = 10,000,000   y = 1     -> 10,000,000
float:  x = 30,000,000   y = 1     -> does not terminate
float:  x =    200,000   y = 0.001 -> does not terminate
bfloat: x =        256   y = 1     -> does not terminate !!
```

Catastrophic Cancellation (case 2)

Let's solve a quadratic equation:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$x^2 + 5000x + 0.25 \qquad x_{1,2} = 0.00005, -5000$

```
(-5000 + std::sqrt(5000.0f * 5000.0f - 4.0f * 1.0f * 0.25f)) / 2
(-5000 + std::sqrt(25000000.0f - 1.0f)) / 2 // !!
(-5000 + std::sqrt(25000000.0f)) / 2
(-5000 + 5000) / 2 = 0
```

relative error: $\dfrac{|0 - 0.00005|}{0.00005} = 100\%$

Minimize Error Propagation

- Prefer **multiplication/division** rather than addition/subtraction
- Scale by a **power of two** is safe
- Try to reorganize the computation to **keep near** numbers with the same scale (e.g. sorting numbers)
- Consider to **put a zero** very small number (under a threshold). Common application: iterative algorithms
- **Switch to log scale**. Multiplication becomes Add, and Division becomes Subtraction

References

Suggest reading:

- `What Every Computer Scientist Should Know About Floating-Point Arithmetic`
- `Do Developers Understand IEEE Floating Point?`
- `Yet another floating point tutorial`
- `Unavoidable Errors in Computing`

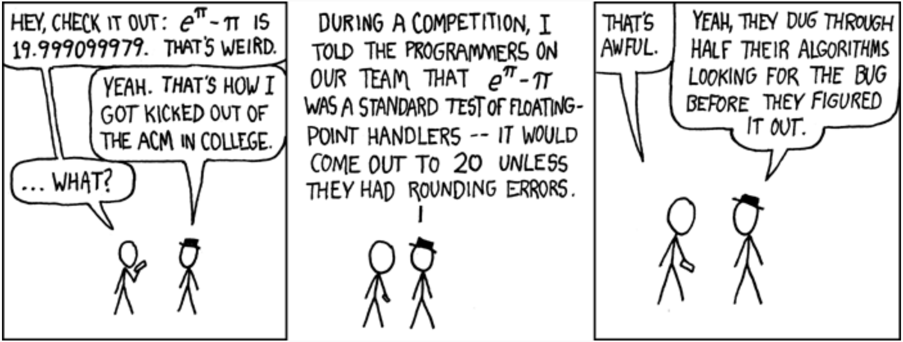
Floating-point Comparison:

- `The Floating-Point Guide - Comparison`
- `Comparing Floating Point Numbers, 2012 Edition`
- `Some comments on approximately equal FP comparisons`
- `Comparing Floating-Point Numbers Is Tricky`

Floating point online visualization tool:

`www.h-schmidt.net/FloatConverter/IEEE754.html`

see "Code Optimization" for other floating-point related issues


HEY, CHECK IT OUT: e^π - π IS 19.999099979. THAT'S WEIRD.
YEAH. THAT'S HOW I GOT KICKED OUT OF THE ACM IN COLLEGE.
... WHAT?
DURING A COMPETITION, I TOLD THE PROGRAMMERS ON OUR TEAM THAT e^π - π WAS A STANDARD TEST OF FLOATING-POINT HANDLERS -- IT WOULD COME OUT TO 20 UNLESS THEY HAD ROUNDING ERRORS.
THAT'S AWFUL.
YEAH, THEY DUG THROUGH HALF THEIR ALGORITHMS LOOKING FOR THE BUG BEFORE THEY FIGURED IT OUT.